Solving Linear homogeneous recurrence relations

(Given linear recursion formula, find a closed-form solution.)

Fibonacci

$F_n = F_{n-1} + F_{n-2}$

"$F_n = 3F_{n-1} + 7F_{n-2} + (-2)F_{n-3}$

$a_0 = R_1, a_1 = R_2, \ldots, a_{k-1} = R_{k-1}$

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$

Substitute $a_n = r^n$. What equation do you get?

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

divide by r^{n-k}

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} \cdots + c_{k-1} r^1 + c_k$$

$\frac{r^n}{r^{n-k}} = r^k$ $\quad n-(n-k)$

$\frac{r^{n-1}}{r^{n-k}} = r^{n-1-(n-k)}$

Characteristic Equation

$$\Rightarrow r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

A polynomial equation

like $x^3 - 3x^2 - x + 2 = 0$

Typically, I will get k solutions $r_1, r_2, \ldots r_k$ to this equation.

Set $a_0 = r_1^0 = 1, a_1 = r_1, a_2 = r_1^2, \ldots, a_{k-1} = r_1^{k-1}$

$r_1^k = a_k = c_1 r_1^{k-1} + c_2 r_1^{k-2} \cdots + c_k$

$= c_1 a_{k-1} + c_2 a_{k-2} \cdots + c_k$ — works

So $a_n = r_1^n$ does solve the recurrence formula (for a certain set of initial conditions)

Similarly $a_n = r_2^n$ works

$\vdots$

$a_n = r_k^n$ works.

Because the recurrence formula is <u>linear</u> and <u>homogeneous</u>, if we add two solutions $a_n = f(n)$, $a_n = g(n)$, we get another solution $a_n = f(n) + g(n)$ — this also works.

Also, we can multiply any solution by a constant, and it still work.

e.g. If $a_n = 3^n$ works, then also $a_n = (17)(3^n)$ works.

$\therefore$ The recurrence formula

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

has a solution of the form

$$a_n = \underline{\alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are any constants, and $r_1, r_2, \ldots, r_k$ are the roots of the polynomial equation:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r^1 - c_k = 0.$$

<u>Thm</u>: Every solution to $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$ has the form written above, as long as the roots $r_1, \ldots, r_k$ are distinct.

In fact, if we specify the initial conditions
$a_0 = R_0,\ a_1 = R_1,\ a_2 = R_2, \dots, a_{k-1} = R_{k-1}$, then
there exists unique values of the constants
$\alpha_1, \dots, \alpha_k$ to solve the whole system.
(k equations, k unknowns } linear algebra).

Example. Find the closed form formula for F_n, the n^{th} Fibonacci number $\left[\begin{array}{l} F_1 = 1,\ F_2 = 1 \\ F_n = F_{n-1} + F_{n-2} \end{array}\right]$.
$c_1 = 1 \quad c_2 = 1,\ k = 2$

Characteristic equation:

Try $F_n = r^n$ $\quad r^n = r^{n-1} + r^{n-2}$

divide by r^{n-2} $\Rightarrow$ $r^2 = r + 1$ Characteristic equation

$r^2 - r - 1 = 0$ $\quad ax^2 + bx + c = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

roots $r = \dfrac{1 \pm \sqrt{1+4}}{2}$

$r = \dfrac{1 \pm \sqrt{5}}{2}$ ← roots

$F_n = \left(\dfrac{1+\sqrt{5}}{2}\right)^n$ works $F_n = F_{n-1} + F_{n-2}$

$F_n = \left(\dfrac{1-\sqrt{5}}{2}\right)^n$ works

General Solution

$$F_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

we need to find α_1, α_2 from initial condition.

Initial Condition

$\star$ $1 = F_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^1$

$\star\star$ $1 = F_2 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^2 + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^2$

Note:

- $\frac{1-\sqrt{5}}{1+\sqrt{5}} = \frac{(1-\sqrt{5})(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} = \frac{1+5-2\sqrt{5}}{1-5} = \frac{6-2\sqrt{5}}{-4} = \frac{-3+\sqrt{5}}{2}$
- $(1+\sqrt{5})(1-\sqrt{5}) = 1-5 = -4$
- $\frac{1+\sqrt{5}}{1-\sqrt{5}} = \frac{(1+\sqrt{5})(1+\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})} = \frac{6+2\sqrt{5}}{\underbrace{1-5}_{-4}} = \frac{-3-\sqrt{5}}{2}$
- $(1+\sqrt{5})^2 = 6+2\sqrt{5}$, $(1-\sqrt{5})^2 = 6-2\sqrt{5}$

We multiply the first equation by $(1+\sqrt{5})2$ and the second by 4 to get

$\star$ $2+2\sqrt{5} = \alpha_1(6+2\sqrt{5}) - 4\alpha_2$

$\star\star$ $4 = \alpha_1(6+2\sqrt{5}) + (6-2\sqrt{5})\alpha_2$

We subtract:

$$-2+2\sqrt{5} = 0 + (-10+2\sqrt{5})\alpha_2$$

$$\Rightarrow \alpha_2 = \frac{-2+2\sqrt{5}}{-10+2\sqrt{5}} = \frac{2(\sqrt{5}-1)}{-2(5-\sqrt{5})} = \frac{(\sqrt{5}-1)}{-\sqrt{5}(\sqrt{5}-1)} = \frac{-1}{\sqrt{5}}.$$

Then we plug α_2 back into the original $\star$ equation:

$$1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\Rightarrow \alpha_1 = \left[1 + \frac{1}{\sqrt{5}}\frac{(1-\sqrt{5})}{2}\right]\left(\frac{2}{1+\sqrt{5}}\right)$$

$$\Rightarrow \alpha_1 = \frac{2}{1+\sqrt{5}} + \frac{1}{\sqrt{5}}\left(\frac{-3+\sqrt{5}}{2}\right)$$

$$\Rightarrow \alpha_1 = \frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} + \frac{(\sqrt{5})(-3+\sqrt{5})}{5\cdot 2}$$

$$= \frac{2-2\sqrt{5}}{-4} + \frac{-3\sqrt{5}+5}{10} = \cancel{-\frac{1}{2}} + \frac{\sqrt{5}}{2} - \frac{3}{10}\sqrt{5} + \cancel{\frac{1}{2}}$$

$$= \frac{5\sqrt{5}-3\sqrt{5}}{10} = \frac{2\sqrt{5}}{10} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$$

$\therefore$ The n^{th} Fibonacci number is

$$F_n = \underset{\uparrow \alpha_1}{\frac{1}{\sqrt{5}}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \underset{\uparrow \alpha_2}{\frac{-1}{\sqrt{5}}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Quiz:

1. Give your favorite Halloween costume.
2. Give the first 5 Fibonacci numbers.
3. If $a_{n+1} = a_n - 2$ and $a_1 = 17$, find a_4.
